ON $G_2$-MORPHIC MANIFOLDS AND $G_2$ STRUCTURES

ABSTRACT

In this paper, we consider two $G_2$-morphic 7-manifolds with $G_2$ structures and show that they belong to the same class of $G_2$ structures. The converse may not be true, however.

Keywords: $G_2$ Structures, $G_2$-Morphisms.

$G_2$-MORFİK MANİFOLDLAR VE $G_2$ YAPILARI

ÖZ

Bu çalışmada $G_2$ yapısına sahip 7-boyutlu $G_2$-morfik manifoldlar ele alınmıştır. Herhangi iki manifoldun $G_2$-morfik olması durumunda, bu manifoldların $G_2$ yapılarının aynı sınıfında yer aldıkları gösterilmiştir. Ancak, önermenin tersi doğru olmayabilir.

Anahtar Kelimeler: $G_2$ Yapıları, $G_2$-Morfizmler.
1. INTRODUCTION

Consider $\mathbb{R}^7$ with its standard basis $\{e_1, \ldots, e_7\}$ and dual basis $\{e^1, \ldots, e^7\}$. The 3-form

$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$

where $e^{ijk} = e^i \wedge e^j \wedge e^k$ is called the fundamental 3-form on $\mathbb{R}^7$. The group $G_2$ is defined as

$G_2 : = \{ f \in GL(7, \mathbb{R}) | f^{\star} \varphi_0 = \varphi_0 \}$,

where $GL(7, \mathbb{R})$ is the group of isomorphisms of $\mathbb{R}^7$. The group $G_2$ is a compact, simple and simply connected 14-dimensional Lie group. A 7-dimensional manifold $M$ is called a manifold with $G_2$ structure if its structure group reduces to the group $G_2$. The existence of a $G_2$ structure is equivalent to the existence of a 3-form on $M$ which can be locally written as $\varphi_0$. This 3-form gives a Riemannian metric and a volume form on $M$ (Bryant, 1987).

Manifolds having $G_2$ structures were classified by Fernandez and Gray in (Fernandez and Gray, 1982). There are 16 classes of $G_2$ structures depending on the space the covariant derivative of the fundamental 3-form $\varphi$ belongs to. The defining relations for each of the 16 classes were given by Fernandez and Gray (Fernandez and Gray, 1982) and then an equivalent characterization was obtained by Cabrera using $d\varphi$ and $d^{\star} \varphi$ in (Cabrera, 1996). This characterization is given in the Table 1.

Note that $* d\varphi \wedge \varphi = - * d \star \varphi \wedge \varphi$, $\alpha = - \frac{1}{4} (* d \varphi \wedge \varphi)$, $\beta = - \frac{1}{3} (* d \varphi \wedge \varphi)$ and $f = \frac{1}{7} (\varphi \wedge d\varphi)$ (Cabrera, 1996).

Let $(M_1, \varphi_1)$ and $(M_2, \varphi_2)$ be 7-manifolds with $G_2$ structures. If there exists a diffeomorphism $F: M_1 \rightarrow M_2$ such that $F^{\star}(\varphi_2) = \varphi_1$, then $F$ is called a $G_2$-morphism and $(M_1, \varphi_1)$ and $(M_2, \varphi_2)$ are said to be $G_2$-morphic (Cho, Salur and Todd, 2011).

Let $(M_1, \varphi_1)$ and $(M_2, \varphi_2)$ be $G_2$-morphic. Then $d\varphi_1 = 0$ iff $d\varphi_2 = 0$ since $d$ commutes with pullback maps (Cho, Salur and Todd, 2011).

Table 1. Classification of $G_2$ structures

<table>
<thead>
<tr>
<th>$\mathcal{P}$</th>
<th>$d\varphi = 0$ and $d \star \varphi = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{W}_1$</td>
<td>$d\varphi = k \star \varphi$ and $d \star \varphi = 0$</td>
</tr>
<tr>
<td>$\mathcal{W}_2$</td>
<td>$d\varphi = 0$</td>
</tr>
<tr>
<td>$\mathcal{W}_3$</td>
<td>$d \star \varphi = 0$ and $d\varphi \wedge \varphi = 0$</td>
</tr>
<tr>
<td>$\mathcal{W}_4$</td>
<td>$d\varphi = \alpha \wedge \varphi$ and $d \star \varphi = \beta \wedge \varphi$</td>
</tr>
<tr>
<td>$\mathcal{W}_1 \oplus \mathcal{W}_2$</td>
<td>$d\varphi = k \star \varphi$ and $d \star \varphi \wedge \varphi = 0$</td>
</tr>
<tr>
<td>$\mathcal{W}_1 \oplus \mathcal{W}_3$</td>
<td>$d \star \varphi = 0$</td>
</tr>
<tr>
<td>$\mathcal{W}_2 \oplus \mathcal{W}_3$</td>
<td>$d\varphi \wedge \varphi = 0$ and $d \star \varphi \wedge \varphi = 0$</td>
</tr>
<tr>
<td>$\mathcal{W}_1 \oplus \mathcal{W}_4$</td>
<td>$d\varphi = \alpha \wedge \varphi + f \star \varphi$ and $d \star \varphi = \beta \wedge \varphi$</td>
</tr>
<tr>
<td>$\mathcal{W}_2 \oplus \mathcal{W}_4$</td>
<td>$d\varphi = \alpha \wedge \varphi$</td>
</tr>
<tr>
<td>$\mathcal{W}_3 \oplus \mathcal{W}_4$</td>
<td>$d\varphi \wedge \varphi = 0$ and $d \star \varphi = \beta \wedge \varphi$</td>
</tr>
<tr>
<td>$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$</td>
<td>$* d\varphi \wedge \varphi = 0$ or $d \star \varphi \wedge \varphi = 0$</td>
</tr>
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</tr>
<tr>
<td>$\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$</td>
<td>$d\varphi \wedge \varphi = 0$</td>
</tr>
<tr>
<td>$\mathcal{W}$</td>
<td>No relation</td>
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</tbody>
</table>
In the present paper, we show that $G_2$-morphisms do not only preserve closed $G_2$-structures, but also other 15 classes too. We consider two $G_2$-morphic manifolds and prove that both belong to the same class of $G_2$-structures.

2. RELATIONS BETWEEN $G_2$-MORPHIC MANIFOLDS

Take two $G_2$-morphic manifolds $(M_1, \varphi_1)$ and $(M_2, \varphi_2)$. Then there exists a diffeomorphism $F: M_1 \rightarrow M_2$ such that $F^*(\varphi_2) = \varphi_1$. Let $g_1$ and $g_2$ denote the Riemannian metrics and $\Omega_1, \Omega_2$ the volume forms determined by $\varphi_1$ and $\varphi_2$ respectively. Then for $x, y \in TM_1$ and $x', y' \in TM_2$, followings hold (Bryant, 1987):

$$(x \lrcorner \varphi_1) \wedge (y \lrcorner \varphi_1) \wedge \varphi_1 = g_1(x, y)\Omega_1,$$

$$(x' \lrcorner \varphi_2) \wedge (y' \lrcorner \varphi_2) \wedge \varphi_2 = g_2(x', y')\Omega_2,$$

where the symbol "\lrcorner" denotes the contraction of the 3-form $\varphi$. Now

$$g_1(x, y)\Omega_1 = (x \lrcorner F^*\varphi_2) \wedge (y \lrcorner F^*\varphi_2) \wedge F^*\varphi_2$$

$$= (F^*(F(x)\lrcorner \varphi_2)) \wedge (F^*(F(y)\lrcorner \varphi_2)) \wedge F^*\varphi_2$$

$$= F^*\{g_2(F(x), F(y))\Omega_2\}$$

$$= g_2(F(x), F(y))F^*\Omega_2.$$

Take a local orthonormal frame $\{e_1, ..., e_7\}$ on an open neighbourhood of $q \in M_1$. Then

$$g_1(x, y)\Omega_1 = g_2(F(x), F(y))F^*\Omega_2(e_1, ..., e_7)$$

$$= g_2(F(x), F(y))\Omega_2(F(e_1), ..., F(e_7))$$

$$= g_2(g_2(F(x), F(y))(det g_2(F(e_i), F(e_j)))^{1/2}.$$

Thus if $(M_1, \varphi_1)$ and $(M_2, \varphi_2)$ are $G_2$-morphic, then

$$kF^*g_2 = g_1$$

where $k = (det g_2(F(e_i), F(e_j)))^{1/2}.$

Let $P_1$ and $P_2$ denote the two-fold vector cross products determined by $\varphi_1$ and $\varphi_2$, respectively. Then

$$g_2(P_2(F(x), F(y), F(z)) = \varphi_2(F(x), F(y), F(z))$$

$$= \varphi_1(x, y, z)$$

$$= g_1(P_1(x, y))$$

$$= g_2(kF_2(P_1(x, y)), F(z)).$$
for all \( x, y \in T_q M_1 \). Since \( F \) is an isomorphism and \( g_2 \) is non-degenerate, we obtain:

\[
F^* P_2 = k F \circ P_1.
\]

Note that the isomorphism \( F^* : TM_1 \rightarrow TM_2 \) induces an isomorphism

\[
F^* : \Lambda^p (TM_2)^* \rightarrow \Lambda^p (TM_1)^*.
\]

Now we extend the metrics \( g_1 \) and \( g_2 \) to spaces \( \Lambda^p (TM_1)^* \) and \( \Lambda^p (TM_2)^* \), respectively.

Let \( \alpha \) and \( \beta \) be 1-forms on \( M_2 \). In this case \( F^* \alpha \) and \( F^* \beta \) are 1-forms on \( M_1 \). Let \( \# \) denote the metric dual of a given 1-form or a vector field. Take a local orthonormal frame \( \{ e_1, \ldots, e_7 \} \) on an open neighbourhood of a point \( q \) of \( M_1 \). Then for any vector \( x \in T_{F(q)} M_2 \), we have

\[
\alpha(x) = g_2(x, \alpha^\#)
\]

\[
= g_2(F((F)^{-1}(x)), F((F)^{-1}(\alpha^\#)))
\]

\[
= k^{-1} g_1((F)^{-1}(x), (F)^{-1}(\alpha^\#))
\]

\[
= k^{-1} ((F)^{-1}(\alpha^\#))^\# ((F)^{-1}(x))
\]

and thus

\[
(F^* \alpha)^\# = k^{-1}(F)^{-1}(\alpha^\#).
\]

This gives

\[
g_1(F^* \alpha, F^* \beta) = g_1((F^* \alpha)^\#, (F^* \beta)^\#)
\]

\[
= k^{-2} g_1((F)^{-1}(\alpha^\#), (F)^{-1}(\beta^\#))
\]

\[
= k^{-2} k g_2(\alpha^\#, \beta^\#)
\]

\[
= k^{-1} g_2(\alpha, \beta),
\]

that implies

\[
g_2(\alpha, \beta) = k g_1(F^* \alpha, F^* \beta)
\]

for any 1-forms \( \alpha, \beta \) on \( M_2 \).

Now let \( \alpha = \alpha_1 \wedge \ldots \wedge \alpha_p \) and \( \beta = \beta_1 \wedge \ldots \wedge \beta_p \) be \( p \)-forms on \( M_2 \). Then

\[
F^*(\alpha) = F^*(\alpha_1) \wedge \ldots \wedge F^*(\alpha_p) \quad \text{and} \quad F^*(\beta) = F^*(\beta_1) \wedge \ldots \wedge F^*(\beta_p)
\]

are \( p \)-forms on \( M_1 \). Then

\[
g_1(F^*(\alpha), F^*(\beta)) = g_1(F^*(\alpha_1) \wedge \ldots \wedge F^*(\alpha_p), F^*(\beta_1) \wedge \ldots \wedge F^*(\beta_p))
\]

\[
= \det g_1(F^*(\alpha_i), F^*(\beta_j)) = k^{-p} \det g_2(\alpha_i, \beta_j)
\]

\[
= k^{-p} g_2(\alpha_1 \wedge \ldots \wedge \alpha_p, \beta_1 \wedge \ldots \wedge \beta_p) = k^{-p} g_2(\alpha, \beta),
\]

which gives,

\[
g_2(\alpha, \beta) = k^p g_1(F^* \alpha, F^* \beta)
\]
for p-forms \( \alpha, \beta \) on \( M_2 \). Now since 
\[
k^2 = \det g_2(F_*(e_i), F_*(e_j)) \quad (1)
\]
we have
\[
k^2 = \det g_2((F_*(e_i))^\flat, (F_*(e_j))^\flat) = \det(k^{-1}g_1(e_i, e_j)) = k^{-7}g_1(e_1 \wedge ... \wedge e_7, e_1 \wedge ... \wedge e_7) = k^{-7}.
\]
Thus we obtain \( k^9 = 1 \) which is possible only if \( k = 1 \).

Therefore if \((M_1, \varphi_1)\) and \((M_2, \varphi_2)\) are \( G_2 \)-morphic, then
\[
F^* g_2 = g_1, \quad (1)
\]
\[
F^* P_2 = F_\circ P_1. \quad (2)
\]

Conversely, if (1) and (2) hold, then \((M_1, \varphi_1)\) and \((M_2, \varphi_2)\) are \( G_2 \)-morphic.

Let \(*_1\) and \(*_2\) denote Hodge-star operators determined by metrics \( g_1 \) and \( g_2 \), respectively. If \( \alpha \) and \( \beta \) are p and 7-p forms on \( M_2 \), then \( F^*\alpha \) and \( F^*\beta \) are p and 7-p forms on \( M_1 \). Take a local orthonormal frame \( \{e_1, ..., e_7\} \) on an open neighbourhood of a point \( q \) of \( M_1 \). We compute the following:

\[
g_2(*_2 \alpha, \beta) = g_2(\alpha \wedge \beta, \Omega_2)
\]
\[
= g_2(F^*(\alpha \wedge \beta), F^*\Omega_2) = g_2(F^*\alpha \wedge F^*\beta, mg_1)
\]
\[
= m g_1(_, F^*\beta)
\]
\[
= m g_1(F^*((F^*)^{-1}(\alpha)), F^*((F^*)^{-1}(F^*\beta))) = m g_2((F^*)^{-1}(\alpha), \beta).
\]

where \( m = (F^*\Omega_2)(e_1, ..., e_7) \). Therefore

\[
F^*(*_2 \alpha) = m *_1 F^*\alpha.
\]

Now we use the formula \(* \varphi(w, x, y, z) = \frac{1}{3} g(w, \Xi_{xyz} P(P(x, y), z)) \) given in (Fernandez and Gray, 1982) for the Hodge-star \(*\) of the fundamental 3-form \( \varphi \) of a manifold with \( G_2 \) structure. There exist \( w, x, y, z \in T_q M_1 \) such that \( F_*(w) = w', F_*(x) = x', F_*(y) = y', F_*(z) = z' \) for any \( w', x', y', z' \in T_{F(q)} M_2 \). Thus
\[ *_{2} \varphi_{2}(w', x', y', z') = *_{2} \varphi_{2}(F_{s}(w), F_{s}(x), F_{s}(y), F_{s}(z)) \]
\[ = \frac{1}{3} g_{2}(F_{s}(w), \Sigma_{xyz} P_{2}(P_{2}(F_{s}(x), F_{s}(y)), F_{s}(z))) \]
\[ = \frac{1}{3} g_{2}(F_{s}(w), \Sigma_{xyz} P_{2}(F_{s}(P_{1}(x, y)), F_{s}(z))) \]
\[ = \frac{1}{3} g_{2}(F_{s}(w), \Sigma_{xyz} P_{1}(P_{1}(x, y), z)) \]
\[ = \frac{1}{3} g_{1}(w, \Sigma_{xyz} P_{1}(P_{1}(x, y), z)) \]
\[ = *_{1} \varphi_{1}(w, x, y, z), \]

i.e. we get
\[ F^{*}(\varphi_{2}) = *_{1} \varphi_{1}. \]

Put \( \alpha = \varphi_{2} \) in the equation \( F^{*}(\varphi_{2}) = m *_{1} F^{*} \alpha \), to observe that \( m = 1 \). Hence we get the equation
\[ F^{*}(\varphi_{2}) = *_{1} F^{*} \alpha \quad (3) \]
for any p-form \( \alpha \) on \( M_{2} \).

### 3. \( G_{2} \)-MORPHIC MANIFOLDS AND CLASSES OF \( G_{2} \) STRUCTURES

Let \((M_{1}, \varphi_{1})\) and \((M_{2}, \varphi_{2})\) be \( G_{2} \)-morphic manifolds. There exists a diffeomorphism \( F: M_{1} \rightarrow M_{2} \) such that \( F^{*}(\varphi_{2}) = \varphi_{1} \). Assume \( \mathcal{U} \) is one of the sixteen classes of \( G_{2} \) structures. In this section, we show that \( M_{2} \in \mathcal{U} \) iff \( M_{2} \in \mathcal{U} \). We use the characterization of Cabrera in (Cabrera, 1996).

We investigate each class separately. We use relations (1), (2) and (3) we found in the previous section.

**The class \( \mathcal{P} \):** Let \( M_{1} \in \mathcal{P} \). Then \( d \varphi_{1} = 0 \) and \( d *_{1} \varphi_{1} = 0 \). Thus
\[ 0 = d \varphi_{1} = d F^{*} \varphi_{2} = F^{*} d \varphi_{2} \]
and \( 0 = d *_{1} \varphi_{1} = d (F^{*}(\varphi_{2})) = F^{*}(d *_{2} \varphi_{2}) \). Since \( F^{*} \) is an isomorphism, we get \( d \varphi_{2} = 0 \) and \( d *_{2} \varphi_{2} = 0 \).

**The class \( \mathcal{W}_{1} \):** Let \( M_{1} \in \mathcal{W}_{1} \). Then \( d \varphi_{1} = t *_{1} \varphi_{1} \) and \( d *_{1} \varphi_{1} = 0 \) for \( d \varphi_{1} \neq 0 \). It is enough to show the first condition. For \( t \neq 0 \) we have \( d \varphi_{1} = t *_{1} \varphi_{1} \), which is equivalent to
\[ d(F^{*} \varphi_{2}) = t(F^{*}(\varphi_{2})). \]
Since \( d \) commutes with pullback maps, we have \( F^{*}(d \varphi_{2}) = F^{*}(t *_{2} \varphi_{2}) \). This implies \( d \varphi_{2} = t *_{2} \varphi_{2} \) since \( F^{*} \) is one-to-one. Since \( d \varphi_{1} \neq 0 \) and \( F^{*} \) is an isomorphism, we get \( d \varphi_{2} \neq 0 \). Thus \( M_{2} \notin \mathcal{P} \).

**The class \( \mathcal{W}_{2} \):** Let \( M_{1} \in \mathcal{W}_{2} \). Then \( d \varphi_{1} = 0 \) for \( d *_{1} \varphi_{1} \neq 0 \). Here it is enough to see that \( M_{2} \) can not belong to the class \( \mathcal{P} \). Assume that \( M_{2} \in \mathcal{P} \). Then \( d *_{2} \varphi_{2} = 0 \) which implies that
\[ d((F^{*})^{-1}(t *_{1} \varphi_{1})) = (F^{*})^{-1}(d *_{1} \varphi_{1}). \]
Thus \( d *_{1} \varphi_{1} = 0 \) which is a contradiction. Thus \( M_{2} \notin \mathcal{P} \).
The class $\mathcal{W}_3$: Let $M_1 \in \mathcal{W}_3$. Then $d \ast_1 \varphi_1 = 0$ and $d \varphi_1 \wedge \varphi_1 = 0$ for $d \varphi_1 \neq 0$. It is enough to see the second condition. Since

$$0 = d \varphi_1 \wedge \varphi_1 = d(F^* \varphi_2) \wedge (F^* \varphi_2) = F^*(d \varphi_2) \wedge (F^* \varphi_2) = F^*(d \varphi_2 \wedge \varphi_2)$$

and $F^*$ is an isomorphism, we get $d \varphi_2 \wedge \varphi_2 = 0$. We can use the arguments we used while showing $M_2 \not\in \mathcal{P}$ in the previous class here to see that $M_2 \not\in \mathcal{P}$ similarly.

The class $\mathcal{W}_4$: Let $M_1 \in \mathcal{W}_4$. Then $d \varphi_1 = \alpha \wedge \varphi_1$ and $d \ast_1 \varphi_1 = \beta \wedge \ast_1 \varphi_1$ for $\alpha, \beta \neq 0$. The condition $d \varphi_1 = \alpha \wedge \varphi_1$ is equivalent to $d(F^* \varphi_2) = \alpha \wedge F^* \varphi_2$. We can write this equation as $F^*(d \varphi_2) = F^*((F^*)^{-1} \alpha) \wedge F^* \varphi_2 = F^*((F^*)^{-1} \alpha \wedge \varphi_2)$ and since $F^*$ is an isomorphism, we have $d \varphi_2 = (F^*)^{-1} \alpha \wedge \varphi_2$. In addition, $d \ast_1 \varphi_1 = \beta \wedge \ast_1 \varphi_1$ means $d(F^* \ast_2 \varphi_2) = \beta \wedge F^* \ast_2 \varphi_2$. Hence

$$F^*(d \ast_2 \varphi_2) = F^*((F^*)^{-1} \beta) \wedge F^* \ast_2 \varphi_2 = F^*((F^*)^{-1} \beta \wedge \ast_2 \varphi_2)$$

and similarly we get

$$d \ast_2 \varphi_2 = (F^*)^{-1} \beta \wedge \ast_2 \varphi_2.$$ Note also that $d \varphi_2 \neq 0$ and $d \ast_2 \varphi_2 \neq 0$. Thus $M_2 \in \mathcal{W}_4$.

The class $\mathcal{W}_1 \oplus \mathcal{W}_2$: Let $M_1 \in \mathcal{W}_1 \oplus \mathcal{W}_2$. Then $d \varphi_1 = t \ast_1 \varphi_1$ and

$$(\ast_1 d \ast_1 \varphi_1) \wedge \ast_1 \varphi_1 = 0 \text{ for } d \varphi_1 \neq 0 \text{ and } d \ast_1 \varphi_1 \neq 0.$$ It is enough to show the following:

$$0 = (\ast_1 d \ast_1 \varphi_1) \wedge \ast_1 \varphi_1$$

$$= \ast_1 d(F^* \ast_2 \varphi_2) \wedge F^*(\ast_2 \varphi_2)$$

$$= \ast_1 F^*(d \ast_2 \varphi_2) \wedge F^*(\ast_2 \varphi_2)$$

$$= F^*(d \ast_2 \varphi_2) \wedge F^*(\ast_2 \varphi_2)$$

$$= F^*((\ast_2 d \ast_2 \varphi_2) \wedge \ast_2 \varphi_2).$$

This gives $(\ast_2 d \ast_2 \varphi_2) \wedge \ast_2 \varphi_2 = 0$. Similar to previous classes we can see that $M_2$ can not belong to subclasses of $\mathcal{W}_1 \oplus \mathcal{W}_2$.

The class $\mathcal{W}_1 \oplus \mathcal{W}_3$: Let $M_1 \in \mathcal{W}_1 \oplus \mathcal{W}_3$. Then $d \ast_1 \varphi_1 = 0$ for $d \varphi_1 \neq 0$. Here it is enough to show that $M_2$ can not be an element of subclasses of $\mathcal{W}_1 \oplus \mathcal{W}_3$ and this can be seen again by using that $F^*$ is an isomorphism.

The class $\mathcal{W}_2 \oplus \mathcal{W}_3$: Let $M_1 \in \mathcal{W}_2 \oplus \mathcal{W}_3$. Then $d \varphi_1 \wedge \varphi_1 = 0$ and $(\ast_1 d \varphi_1) \wedge \varphi_1 = 0$ for $d \varphi_1 \neq 0$ and $d \ast_1 \varphi_1 \neq 0$. It is enough to see followings:

$$0 = (\ast_1 d \varphi_1) \wedge \varphi_1$$

$$= \ast_1 d(F^* \varphi_2) \wedge F^* \varphi_2$$

$$= \ast_1 F^*(d \varphi_2) \wedge F^* \varphi_2$$

$$= F^*(d \varphi_2) \wedge F^* \varphi_2$$

$$= F^*((\ast_2 d \varphi_2) \wedge \varphi_2).$$

This implies $(\ast_2 d \varphi_2) \wedge \varphi_2 = 0$. We can also show similarly that $M_2$ can not be in the subclasses.
The class $\mathcal{W}_1 \oplus \mathcal{W}_4$: Let $M_1 \in \mathcal{W}_1 \oplus \mathcal{W}_4$. Then $d\varphi_1 = \alpha \wedge \varphi_1 + f \ast_1 \varphi_1$ and $d \ast_1 \varphi_1 = \beta \wedge \ast_1 \varphi_1$ for a nonzero function $f$, $d\varphi_1 \neq 0$ and $d \ast_1 \varphi_1 \neq 0$. We should see the following:

$$d\varphi_1 = \alpha \wedge \varphi_1 + f \ast_1 \varphi_1,$$

$$d(F^*\varphi_2) = \alpha \wedge F^*\varphi_2 + f F^*(\ast_2 \varphi_2),$$

$$F^*(d\varphi_2) = F^*((F^*)^{-1}\alpha) \wedge F^*\varphi_2 + f F^*(\ast_2 \varphi_2) = F^*((F^*)^{-1}\alpha \wedge \varphi_2 + f \circ F^{-1} \ast_2 \varphi_2),$$

which implies $d\varphi_2 = (F^*)^{-1}\alpha \wedge \varphi_2 + f \circ F^{-1} \ast_2 \varphi_2$. We can also show similarly that $M_2$ cannot belong to the subclasses.

The class $\mathcal{W}_2 \oplus \mathcal{W}_4$: Let $M_1 \in \mathcal{W}_2 \oplus \mathcal{W}_4$. Then $d\varphi_1 = \alpha \wedge \varphi_1$ for $d\varphi_1 \neq 0$. We showed that the condition $d\varphi_1 = \alpha \wedge \varphi_1$ is equivalent to the condition $d\varphi_2 = (F^*)^{-1}\alpha \wedge \varphi_2$. We can see that $M_2$ cannot belong to the subclasses similar to previous classes.

The class $\mathcal{W}_3 \oplus \mathcal{W}_4$: Let $M_1 \in \mathcal{W}_3 \oplus \mathcal{W}_4$. Then $d\varphi_1 \wedge \varphi_1 = 0$ and $d \ast_1 \varphi_1 = \beta \wedge \ast_1 \varphi_1$ for $d\varphi_1 \neq 0$ and $d \ast_1 \varphi_1 \neq 0$. We saw that this is only possible if $d\varphi_2 \wedge \varphi_2 = 0$ and $d \ast_2 \varphi_2 = (F^*)^{-1}\beta \wedge \ast_2 \varphi_2$. We can eliminate the subclasses of $\mathcal{W}_3 \oplus \mathcal{W}_4$ similarly.

The class $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$: Let $M_1 \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$. Then $d \ast_1 \varphi_1 \wedge \ast_1 \varphi_1 = 0$ for $d\varphi_1 \neq 0$ and $d \ast_1 \varphi_1 \neq 0$. If $d \ast_1 \varphi_1 = 0$, then $0 = d\varphi_1 = d(F^*\varphi_2) = F^*d\varphi_2 = F^*(\ast_2 \varphi_2)$, so we get $\ast_2 \varphi_2 = 0$. If $d \ast_1 \varphi_1 \wedge \ast_1 \varphi_1 = 0$, then $0 = d \ast_1 \varphi_1 \wedge \ast_1 \varphi_1$

$$= d(F^*(\ast_2 \varphi_2)) \wedge F^*\varphi_2$$

$$= F^*\ast_2 d(\ast_2 \varphi_2) \wedge F^*\varphi_2$$

$$= F^*(\ast_2 \varphi_2) \wedge F^*(\ast_2 \varphi_2)$$

and this implies $\ast_2 \varphi_2 \wedge F^*\varphi_2 = 0$ since $F^*$ is an isomorphism. We can see that $M_2$ cannot belong to the subclasses similarly.

The class $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$: Let $M_1 \in \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$. Then $d\varphi_1 = \alpha \wedge \varphi_1 + f \ast_1 \varphi_1$ for a non-zero function $f$ and $d\varphi_1 \neq 0$. It is enough to see that $M_2$ is not an element of a subclass. This can be done by using that $F^*$ is an isomorphism.

The class $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$: Let $M_1 \in \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$. Then $d \ast_1 \varphi_1 = \beta \wedge \ast_1 \varphi_1$ for $d\varphi_1 \neq 0$ and $d \ast_1 \varphi_1 \neq 0$. It is enough to see that $M_2$ is not an element of a subclass. This can be done by using that $F^*$ is an isomorphism.
The class $\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$: Let $M_1 \in \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$. Then $d\varphi_1 \wedge \varphi_1 = 0$ for $d\varphi_1 \neq 0$. It is enough to see that $M_2$ is not an element of a subclass. This can be done by using that $F^*$ is an isomorphism.

The class $\mathcal{W}$: Let $M_1 \in \mathcal{W}$. Then $M_2$ is in the same class, too. All subclasses can be eliminated by using that $F^*$ is an isomorphism.

4. CONCLUSION

We observe that for two $G_2$-morphic manifolds $(M_1, \varphi_1)$ and $(M_2, \varphi_2)$, if $M_1$ belongs to $\mathcal{U}$, then $M_2$ is in the same class too. We can easily see that being $G_2$-morphic is an equivalence relation. Thus $M_1 \in \mathcal{U}$ iff $M_2 \in \mathcal{U}$. That is, the classes of $G_2$ structures are preserved under $G_2$-morphisms. The converse may not be true. That is, if there are two diffeomorphic manifolds which are in the same class of $G_2$ structures, they need not to be $G_2$-morphic. An example of two diffeomorphic manifolds which are not $G_2$-morphic are the Aloff-Wallach spaces. These are spaces $M_{k,l} = SU(3)/U(1)_{k,l}$ for $k \neq \pm l$, where $U(1)_{k,l}$ is the subgroup of $SU(3)$ generated by the elements of the form $e^{i(k,l,i(-k-l))}$. Note that a diffeomorphism $G: M_1 \rightarrow M_2$ between Riemannian manifolds $(M_1, g_1)$ and $(M_2, g_2)$ is called a homothety iff $G^*g_2 = cg_1$ for some nonzero constant $c$ (O’Neill, 1983). It is known that $M_{k,l}$ admits two non-homothetic $G_2$-structures in $\mathcal{W}_1$ (Cabrera, Monar and Swann, 1996) and since these $G_2$-structures are not homothetic, they are not $G_2$-morphic. Hence, the classification of manifolds with structure group $G_2$ according to $G_2$-morphisms is finer than the classification of Fernandez and Gray in (Fernandez and Gray, 1982).

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